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CONSERVATIVE AUTONOMOUS NONLINEAR

DIFFERENTIAL SYSTEMS WITH SYMMETRIES

A THESIS

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CONSERVATIVE AUTONOMOUS NONLINEAR

DIFFERENTIAL SYSTEMS WITH SYMMETRIES

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CHAPTER I

INTRODUCTION

The theory of free oscillations of conservative linear systems is well developed, and many nontrivial problems involving such systems are easy to solve. The free oscillations of conservative nonlinear systems are much less well understood, and many problems which one might expect to be tractable prove to be almost impossibly difficult to handle. Between these two extremes lies a middle ground involving the free oscillations of conservative systems which have three properties:

(1) they contain at least one nonlinear element;

(2) they therefore exhibit properties distinctly different from linear systems;

(3) they are mathematically tractable in the sense that the associated differential equations can be decoupled by a suitable linear transformation and solved completely in terms of tabulated functions. The object of this study is to investigate several such systems.

The physical systems considered involve the rectilinear motion of masses coupled by linear and nonlinear springs. Systems of one, two, three and four degrees of freedom are discussed.

Consider now a mechanical spring with force function f. That is, suppose that the force required to distend the spring x inches from its natural length is $f(x)$. Assume that f is continuous and odd and that there is an $x_1 > 0$ such that $f(x) > 0$ for $0 < x < x_1$. Suppose further that f can be expanded in a McLaurin series with radius of convergence

$R > x_1$. Then since f is assumed odd, only odd powers of x are present in the expansion, which has the form

$$f(x) = \sum_{i=0}^{\infty} f^{(2i+1)}(0) \frac{x^{2i+1}}{(2i+1)!} .$$

For sufficiently small values of x, f may be closely approximated by the first two non-zero terms of the expansion; that is,

$$f(x) \approx f'(0)\, x + f'''(0) \frac{x^3}{6}$$

$$= \alpha x + \beta x^3 .$$

Since $f(x) > 0$ for $0 < x < x_1$, it follows that $f'(0) > 0$, $\alpha > 0$. All nonlinear springs considered in this thesis are assumed to be of this cubic nature.

Although the systems considered involve only rectilinear motion of coupled masses, it should be clear that the differential equations of motion could equally well arise from other physical configurations -- for example, a pendulum oscillating with slightly larger than "small" oscillations. See also Cunningham [1], Stoker [2], McLachlan [12].

It should perhaps be noted that the systems of two, three, and four degrees of freedom under consideration possess a large degree of physical symmetry (see Figure 5). This characteristic seems to be significant in finding the decoupling transformations for the equations of motion. As will be seen later, the configurations also possess a less restrictive type of symmetry discussed by Heinbockel and Struble [3].

CHAPTER II

GENERAL PROPERTIES OF THE BASIC NONLINEAR EQUATION

Consider the frictionless mass-spring system pictured schematically in Figure 1(a), where the spring is assumed to be nonlinear and cubic. That is, suppose the force F required to distend the spring x inches from its equilibrium position is given by

$$F = F(x) = \alpha x + \beta x^3 , \tag{1}$$

$\alpha > 0$. Then the equation of motion of the system is readily seen to be

$$m\ddot{x} + \alpha x + \beta x^3 = 0 . \quad (\cdot \sim \frac{d}{dt}) \tag{2}$$

Division of (2) by m yields

$$\ddot{x} + \frac{\alpha}{m} x + \frac{\beta}{m} x^3 = 0$$

or

$$\ddot{x} + ax + bx^3 = 0, \tag{3}$$

thus reducing the number of system parameters from three to two. If desired, the number of system parameters can be reduced to one by the change of variables $\tau = \sqrt{a}\, t$. Then

$$\frac{d^2x}{d\tau^2} + x + \frac{b}{a} x^3 = 0$$

or

$$\frac{d^2x}{d\tau^2} + x + cx^3 = 0 .$$

It is thus clear that in any study of equation (2) it may be assumed that $m = \alpha = 1$ without loss of generality. However, it will later prove to be more convenient merely to assume a unit mass or, equivalently, to assume that equation

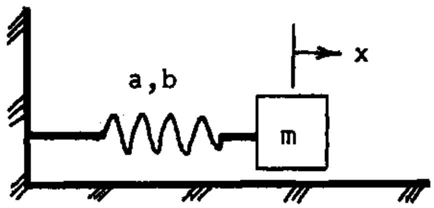


Figure 1(a). Mass-spring System with One Degree of Freedom.

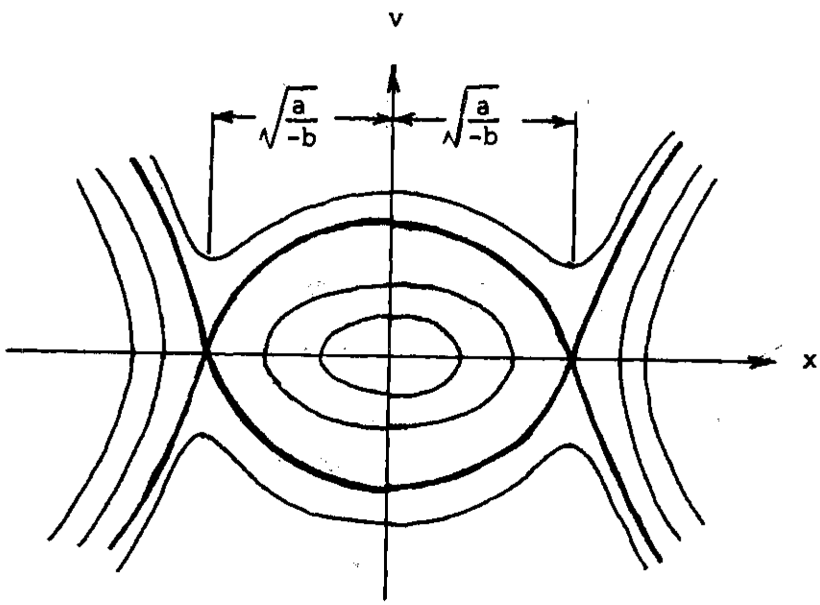


Figure 1(b). Phase-plane Trajectories for the Soft-Spring System.

(2) has been reduced to the form of (3).

If the new variable $v = \dot{x}$ is introduced, the original second-order equation (3) is reduced to the first-order vector equation

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ -ax - bx^3 \end{pmatrix}. \tag{4}$$

Existence and uniqueness of solutions of (4) through any point $(t_0,(x_0,v_0)) \in E_1 \times E_2$ may be guaranteed by reference to Lefschetz [4], Chapter II, Section 2. For simplicity, $t_0 = 0$ is assumed henceforth.

Since the force F in (1) is an integrable function depending only on the displacement of the mass, the physical system is conservative. From (3),

$$\dot{x}(\ddot{x} + ax + bx^3) = \ddot{x}\,\dot{x} + ax\dot{x} + bx^3\dot{x} = 0 ,$$

and upon integration

$$\int_0^t (\ddot{x}\,\dot{x} + ax\dot{x} + bx^3\dot{x})\, d\tau = 0$$

or

$$\frac{1}{2}\,\dot{x}^2(t) + \frac{1}{2}\,ax^2(t) + \frac{1}{4}\,bx^4(t) =$$

$$\frac{1}{2}\,\dot{x}^2(0) + \frac{1}{2}\,ax^2(0) + \frac{1}{4}\,bx^4(0) = E_0, \tag{5}$$

an equation which represents, for various constant values of E_0, the constant-energy curves of the system.

In order to discuss these level curves of the system, two cases must be considered for which the curves and the solutions of equation (3) (or (4)) exhibit rather different characteristics. These cases arise according to whether b in (3) is positive or negative. Geometrically, closure of a trajectory not including a critical point is necessary and sufficient for periodicity of the corresponding solution (see Lefschetz

[4], Chapter II, Section 7, (15.1)). In the event that $b > 0$, all the curves are closed and encircle the origin. If $b < 0$, however, as seen in Figure 1(b), only certain values of E_o in (5) will lead to closed curves, hence to periodic solutions.

If b is positive in (3), the spring is said to be "hard" in that the stiffness increases with distension, whereas if b is negative, the spring is said to be "soft," this because of decreasing stiffness. The soft spring might seem at first to be quite unrealistic in that there must exist a point such that further elongation would cause the spring to tend to stretch itself. However, such a "softening" phenomenon seems feasible over a suitably restricted range of displacements, and indeed such real-world phenomena as pendulums, diatomic molecules, and certain non-uniform magnetic fields exhibit the properties of a soft spring (see Cunningham [1], p. 77).

For the hard spring case, the existence of periodic solutions is guaranteed by closure of the phase-plane trajectories. In the soft spring case, the unbounded trajectories correspond to stretching the spring beyond the "point of no return" so that it begins to stretch itself. Let us now consider under what conditions oscillatory motion is feasible for the soft spring.

As was shown above,

$$E(t) = \frac{1}{2} v^2 + \frac{1}{2} ax^2 + \frac{1}{4} bx^4 = E_o \qquad (v = \dot{x})$$

is an integral of (3) corresponding physically to the total energy of the system. The following conditions are to be examined: (1) $E_o < 0$; (2) $E_o > \frac{a^2}{-4b}$; (3) $E_o = \frac{a^2}{-4b}$; (4) $E_o = 0$; (5) $0 < E_o < \frac{a^2}{-4b}$.

Case 1. Suppose $E_o < 0$. Then

$$v^2 = -ax^2 - \frac{bx^4}{2} + 2E_o \, .$$

If periodic motion is to result, there must be some $t_1 > 0$ so that $v(t_1) = 0$. Solving for x^2 when $v = 0$ yields

$$x^2 = \frac{a \pm \sqrt{a^2 + 4bE_o}}{-b} \, .$$

Since $b < 0$, $a^2 + 4bE_o > a^2$ and the positive root must be chosen for x to be real; hence

$$x^2 = \frac{a + \sqrt{a^2 + 4bE_o}}{-b} > \frac{a}{-b} \, ,$$

$$|x| > \sqrt{\frac{a}{-b}} \, .$$

Now $\frac{dx}{dt} = v$; so

$$\frac{dv}{dt} = -ax - bx^3 \, .$$

For all $x > \sqrt{\frac{a}{-b}}$, $\frac{dv}{dt} > 0$; for all $x < -\sqrt{\frac{a}{-b}}$, $\frac{dv}{dt} < 0$. In particular, these inequalities hold for those values of x for which $v = 0$. But now consider

$$\frac{d}{dx}\left|\frac{dv}{dt}\right| = \begin{cases} (-a - bx^2) + |x|(-2bx) & \text{if } x > \sqrt{\frac{a}{-b}} \\ (a + bx^2) + |x|(-2bx) & \text{if } x < -\sqrt{\frac{a}{-b}} \end{cases}$$

$$= \begin{cases} -a - 3bx^2 & \text{if } x > \sqrt{\frac{a}{-b}} \\ a + 3bx^2 & \text{if } x < -\sqrt{\frac{a}{-b}} \end{cases}$$

or

$$\frac{d}{dx}\left|\frac{dv}{dt}\right| > -a + 3a = 2a > 0 \quad \text{if} \quad x > \sqrt{\frac{a}{-b}} ,$$

$$\frac{d}{dx}\left|\frac{dv}{dt}\right| < a - 3a = -2a < 0 \quad \text{if} \quad x < -\sqrt{\frac{a}{-b}} .$$

When $v = 0$ (under which circumstance $|x| > \sqrt{\frac{a}{-b}}$), both $\frac{dv}{dt}$ and $\frac{d}{dx}\left|\frac{dv}{dt}\right|$ have the same sign as x, so that there is no $t_2 > t_1$ so that $v(t_2) = 0$. Hence periodicity is impossible for $E_o < 0$.

Case 2. Suppose $E_o > \frac{a^2}{-4b}$. Again, if a periodic solution to (3) exists, then $v(t_1) = \frac{d}{dt}x(t_1) = 0$ for some $t_1 > 0$. Then

$$v^2 = 2E_o - ax^2 - \frac{bx^4}{2} = 0 ,$$

$$x^2 = \frac{a \pm\sqrt{a^2 + 4bE_o}}{-b} .$$

But now $a^2 + 4bE_o < 0$; so no real solution exists. Again periodicity is precluded.

Case 3. If $E_o = \frac{a^2}{-4b}$, proceed as above and assume periodicity. Then for some $t_1 > 0$, $v(t_1) = 0$; so

$$v^2 = -ax^2 - \frac{bx^4}{2} - \frac{a^2}{2b} = 0$$

or

$$x^2 = \frac{a \pm\sqrt{a^2 - a^2}}{-b} ,$$

$$|x| = \sqrt{\frac{a}{-b}} .$$

Note now that the points $\left(\sqrt{\frac{a}{-b}}\,,\,0\right)$ and $\left(-\sqrt{\frac{a}{-b}}\,,\,0\right)$ are critical points of (4) and that both lie on the trajectory corresponding to $E_o = \frac{a^2}{-4b}$. Reference to Lefschetz [4], Chapter II, Section 7, (13.4), shows that (x, v) can approach either of the above critical points only as $t \to \infty$ or $t \to -\infty$. Hence periodicity in a non-trivial sense is impossible.

Case 4. Suppose $E_o = 0$.

(a) If the initial conditions are $x(0) = 0$, $\dot{x}(0) = 0$, then $\ddot{x}(0) = -ax(0) - bx^3(0) = 0$ and $(0,0)$ is a critical point. As above, periodicity in a non-trivial sense is contradicted.

(b) Suppose $x_o^2 + v_o^2 = x^2(0) + v^2(0) > 0$. For periodicity, $v(t_1) = 0$ for some $t_1 > 0$;

$$v^2 = -ax^2 - \frac{bx^4}{2} = 0 \;.$$

Since the system is conservative, $x(t_1) \neq 0$. Hence

$$x^2 = \frac{2a}{-b} \;;$$

so

$$|x| > \sqrt{\frac{a}{-b}} \,,$$

and the same argument as in Case 1 shows that v can never be zero again, which nullifies the assumption of an oscillatory solution.

Case 5. To exhaust all possible values that the (energy) integral E may take on, now assume $0 < E_o < \frac{a^2}{-4b}$. Suppose further that

$v(t_1) = 0$, $t_1 > 0$, still a necessity for periodicity. Then

$$v^2 = 2E_o - ax^2 - \frac{bx^4}{2} = 0 ,$$

$$x^2 = \frac{a \pm \sqrt{a^2 + 4bE_o}}{-b} ,$$

and either root is positive. However, note that if

$$x^2 = \frac{a + \sqrt{a^2 + 4bE_o}}{-b} ,$$

then $|x| > \sqrt{\frac{a}{-b}}$ when $v = 0$, and reference to Case 1 excludes the possibility of periodicity. But if $v = 0$ when

$$x^2 = \frac{a - \sqrt{a^2 + 4bE_o}}{-b} ,$$

then $|x| < \sqrt{\frac{a}{-b}}$, and periodic motion is not excluded by the above arguments. Indeed, subject to such constraints on the initial conditions, a periodic solution will be exhibited (see Chapter III).

To examine more closely the conditions under which a periodic solution exists in Case 5, define the function $\mathcal{E}$ on E_2 by

$$\mathcal{E}(x, v) = \frac{ax^2}{2} + \frac{bx^4}{4} + \frac{v^2}{2} .$$

Then $\mathcal{E}$ is just the representation of E, the energy function, in terms of x and v (or $\mathring{x}$). Consider now that part of the phase-plane separatrix (Figure 1(b)) for which $x^2 \leq \frac{a}{-b}$. The equation of this part of the separatrix is

$$\mathcal{E}(x, v) = \mathcal{E}\left(\sqrt{\frac{a}{-b}}, 0\right)$$

or

$$\frac{ax^2}{2} + \frac{bx^4}{4} + \frac{v^2}{2} = \frac{a^2}{-4b}, \quad x^2 \leq \frac{a}{-b}.$$

If (x_o, v_o) is an initial point in the phase plane with $0 < \mathcal{E}(x_o, v_o) < \frac{a^2}{-4b}$, then either (x_o, v_o) is inside or outside this closed part of the separatrix and

$$\mathcal{E}(x, v) = \mathcal{E}(x_o, v_o)$$

is the equation of the phase-plane trajectory of the corresponding solution. Hence, since for a conservative system distinct trajectories can never intersect, a trajectory remains forever either inside or outside this restriction of the separatrix according as it initiated inside or outside. So in considering initial conditions such that $0 < E_o < a^2/-4b$, one need only consider whether i) $|x_o| > \sqrt{\frac{a}{-b}}$ or ii) $|x_o| < \sqrt{\frac{a}{-b}}$. If i) is the case, periodicity is impossible, since the initial point (x_o, v_o) is outside the closed part of the separatrix. If ii) holds, the initial point (x_o, v_o) is inside the closed part of the separatrix; and periodicity is assured, as will be seen in the next chapter.

In summary, conclusions that one may reach from qualitative and geometric studies of the mathematical model of the nonlinear spring with cubic characteristic (and to greater or lesser degree for any feasible nonlinear spring) are:

(1) if the spring is hard, periodic motion results regardless of initial conditions;

(2) for the soft spring certain energies for the system lead to unbounded motion; some energies lead to bounded nonperiodic motion; and some energy levels lead to either unbounded or periodic motion, depending on the initial point (x_0, v_0) in the phase plane, and not merely on the initial energy.

It may be noted in passing that for the hard spring and for periodic motion in the case of the soft spring, the trajectories bear much similarity to those for the simple harmonic oscillator. This fact will be touched upon later in Chapter V.

CHAPTER III

SOLUTION OF THE BASIC NONLINEAR EQUATION

In solving the basic nonlinear equation discussed in Chapter II, the models for the hard and soft springs must be considered separately.

The Hard-Spring Oscillator

The differential equation of motion for the hard-spring oscillator is, in two-parameter form,

$$\ddot{x} + ax + bx^3 = 0 \;, \tag{1}$$

$a > 0$, $b > 0$. The equation (1) will be solved first subject to the rather special initial conditions

$$x(0) = A > 0 \;,$$

$$\dot{x}(0) = 0 \,.$$

This solution will then be generalized to arbitrary initial conditions.

From (1),

$$\frac{d^2x}{dt^2} = -\,(ax + bx^3) \;.$$

Multiplying both sides by $2\frac{dx}{dt}$ yields

$$2\,\frac{dx}{dt}\,\frac{d^2x}{dt^2} = -2\,[\,ax(t) + bx^3(t)\,]\,\frac{dx}{dt}$$

which, upon integration, becomes

$$2\int_o^t \frac{dx}{dz}\frac{d^2x}{dz^2}\,dz = -2\int_o^t (ax + bx^3)\,\frac{dx}{dz}\,dz$$

or

$$\left(\frac{dx}{dt}\right)^2 = -\left(ax^2 + \frac{bx^4}{2} - aA^2 - \frac{bA^4}{2}\right),$$

$$\frac{dx}{dt} = \pm\sqrt{a(A^2 - x^2) + \frac{b}{2}(A^4 - x^4)} \;.$$

Since $A > 0$, $\frac{d^2}{dt^2}\,x(0) < 0$, from which it follows, since $\dot{x}(0) = 0$, that $\frac{d}{dt}\,x(0^+) < 0$. Hence, the negative root is chosen. Separation of variables yields

$$dt = \frac{-dx}{\sqrt{a(A^2 - x^2) + \frac{b}{2}(A^4 - x^4)}} \;.$$

Upon integration,

$$t = \int_o^t dz = -\int_A^x \frac{dz}{\sqrt{a(A^2 - z^2) + \frac{b}{2}(A^4 - z^4)}} \;. \qquad (2)$$

Now

$$a(A^2 - x^2) + \frac{b}{2}(A^4 - x^4)$$

$$= a\left\{(A^2 - x^2) + \frac{b}{2a}(A^2 - x^2)(A^2 + x^2)\right\}$$

$$= a\left\{1 + \frac{b}{2a}(x^2 + A^2)\right\}(A^2 - x^2) \;;$$

so from (2) above,

$$t = -\frac{1}{\sqrt{a}} \int_A^x \frac{dz}{\sqrt{\frac{b}{2a}\,(z^2 + A^2 + \frac{2a}{b})\,(A^2 - z^2)}} \,. \tag{3}$$

Now make the substitution

$$z = A \cos y, \quad 0 \leq y \leq \pi \,,$$

$$y = \operatorname{Cos}^{-1} z/A \,.$$

Then (3) becomes

$$t = -\frac{1}{\sqrt{a}} \int_0^{\psi} \frac{-A \sin y \, dy}{\sqrt{\frac{b}{2a}\,(A^2 \cos^2 y + A^2 + \frac{2a}{b})(A^2 - A^2 \cos^2 y)}}$$

$$= \frac{1}{\sqrt{a}} \int_0^{\psi} \frac{A \sin y \, dy}{\sqrt{\frac{b}{2a}\,(A^2 \cos^2 y + A^2 + \frac{2a}{b})(A^2 \sin^2 y)}} \,,$$

where $\psi = \operatorname{Cos}^{-1} \frac{x}{A}$, or

$$t = \frac{1}{\sqrt{a}} \int_0^{\psi} \frac{dy}{\sqrt{\frac{b}{2a}\,(A^2 \cos^2 y + A^2 + \frac{2a}{b})}} \tag{4}$$

since $\sin y \geq 0$ on $[0, \pi]$. By some algebraic manipulation,

$$\frac{b}{2a}\,(A^2 \cos^2 y + A^2 + \frac{2a}{b}) = 1 + \frac{bA^2}{2a} + \frac{bA^2}{2a} \cos^2 y$$

$$= \frac{a + bA^2}{a} - \frac{bA^2}{2a} + \frac{bA^2}{2a} \cos^2 y$$

$$= \frac{a + bA^2}{a} \left[1 - \frac{bA^2}{2(a + bA^2)}\,(1 - \cos^2 y)\right]$$

$$= \frac{a + bA^2}{a} \left[1 - \frac{bA^2}{2(a + bA^2)} \sin^2 y \right]$$

$$= \frac{a + bA^2}{a} \; (1 - \mu \sin^2 y) \; ,$$

where $\mu = \dfrac{bA^2}{2(a + bA^2)}$. Then from (4),

$$t = \frac{1}{\sqrt{a}} \int_0^{\psi} \frac{dy}{\sqrt{\frac{a + bA^2}{a}} \sqrt{1 - \mu \sin^2 y}}$$

or

$$\sqrt{a + bA^2} \;\; t = \int_0^{\psi} \frac{dy}{\sqrt{1 - \mu \sin^2 y}} \,.$$

But the integral here is just a standard elliptic integral of the first kind,

$$F(\psi , \mu) = \int_0^{\psi} \frac{dy}{\sqrt{1 - \mu \sin^2 y}} \; ;$$

and since $x = A \cos \psi$,

$$\begin{aligned} x(t) &= A \cos(\mathrm{am}\, F(\psi , \mu)) \\ &= A \,\mathrm{cn}\, F(\psi , \mu) \\ &= A \,\mathrm{cn}\, (\sqrt{a + bA^2}\; t ; \mu) \\ &= A \,\mathrm{cn}\, (\omega t ; \mu) , \end{aligned}$$

where $\omega = \sqrt{a + bA^2}$.

Hence the solution of the equation

$$\ddot{x} + ax + bx^3 = 0 \tag{1}$$

subject to the initial conditions $x(0) = A > 0$, $\dot{x}(0) = 0$, is

$$x(t) = A \operatorname{cn} (\omega t ; \mu) , \tag{5}$$

ω and μ as above.

Now, to generalize this solution to arbitrary initial conditions, suppose that $x(0) = x_o$, $\dot{x}(0) = v_o$. The solution of (1) subject to these initial conditions is accomplished by a translation of (5) on the time axis.

If a periodic solution of amplitude $A > 0$ is assumed, then from the energy integral, it follows that

$$E_o = \frac{1}{2} a x_{max}^2 + \frac{1}{4} b x_{max}^4 = \frac{1}{2} aA^2 + \frac{1}{4} bA^4$$
$$= \frac{1}{2} a x_o^2 + \frac{1}{4} b x_o^4 + \frac{1}{2} v_o^2 ,$$

since $x = x_{max} = A$ when $v = 0$. This equation can be solved for A^2,

$$A^2 = \frac{-a \pm \sqrt{a^2 + 4bE_o}}{b} .$$

For $A^2 > 0$, the positive root must be chosen; then

$$A^2 = \frac{-a + \sqrt{a^2 + 4bE_o}}{b} .$$

It is easily verified that if $\omega = \sqrt{a + bA^2}$ and $\mu = \dfrac{bA^2}{2(a + bA^2)}$, any function of the form

$$f(t) = A \operatorname{cn} (\omega t + \alpha; \mu) \tag{6}$$

is a solution of (1). Furthermore (6) contains two arbitrary constants A, α; and the initial conditions $x(0) = x_0$, $\dot{x}(0) = v_0$ are satisfied if A and α are adjusted so that

$$\left.\begin{aligned} bA^4 + 2aA^2 - 4E_0 &= 0 \\ A \operatorname{cn} (\alpha; \mu) &= x_0 \end{aligned}\right\} . \tag{7}$$

Since a solution satisfying given initial conditions is known to be unique, the hard spring oscillator is solved completely in terms of arbitrary initial conditions; the solution is of the form

$$x(t) = A \operatorname{cn} (\omega t + \alpha; \mu) ,$$

ω and μ as above, A and α as in (7).

It should be noted for consistency that if $b = 0$, then $\mu = 0$. In this case

$$F(\psi , 0) = \int_0^{\psi} \frac{dy}{\sqrt{1 - 0}} = \psi$$

so that

$$x(t) = A \cos (\operatorname{am} F(\psi , 0)) = A \operatorname{Cos} \sqrt{a}t ,$$

the solution of the linear oscillator equation, as would certainly be expected.

The Soft-Spring Oscillator

Consider now the soft-spring oscillator. This case will be considered only subject to initial conditions which result in periodic motion.

Here the differential equation of motion is still

$$\ddot{x} + ax + bx^3 = 0 , \tag{1}$$

but now $a > 0$, $b < 0$.

Assume for now the initial conditions $x(0) = 0$, $\dot{x}(0) = v_o > 0$.

Upon integration as before, (1) yields

$$\left(\frac{dx}{dt}\right)^2 - v_o^2 = 2\int_o^t \frac{dx}{dz}\frac{d^2x}{dz^2}\,dz$$

$$= 2\int_o^x -(az+bz^3)dz \quad = -ax^2 - \frac{bx^4}{2} ,$$

so that

$$\left(\frac{dx}{dt}\right)^2 = v_o^2 - ax^2 - \frac{bx^4}{2} . \tag{8}$$

Assume a periodic solution of amplitude A such that $0 < A < \sqrt{\frac{a}{-b}}$.

Then since the energy integral

$$E(t) = \frac{v^2}{2} + \frac{ax^2}{2} + \frac{bx^4}{4} = E_o$$

is constant, $E_o = \frac{v_o^2}{2}$; and since $|x(t)| = A$ when $\dot{x}(t) = 0$,

$$E_o = \frac{v_o^2}{2} = \frac{1}{2}\,a\,A^2 + \frac{1}{4}\,b\,A^4 ,$$

whence $v_o^2 = aA^2 + \frac{1}{2}\,bA^4$ and from (8) above,

$$\left(\frac{dx}{dt}\right)^2 = a(A^2 - x^2) + \frac{1}{2}\,b(A^4 - x^4) .$$

Since $v_o > 0$, choose the positive root,

$$\frac{dx}{dt} = \sqrt{a(A^2 - x^2) + \frac{b}{2}(A^4 - x^4)}\ .$$

Separation of variables and integration then yields

$$t = \frac{1}{\sqrt{a}} \int_o^x \frac{dz}{\sqrt{(A^2 - z^2) + \frac{b}{2a}(A^4 - z^4)}}\ .$$

By some manipulation the integrand becomes

$$(A^2 - x^2) + \frac{b}{2a}(A^4 - x^4) = (A^2 - x^2)\left[1 + \frac{b}{2a}(A^2 + x^2)\right],$$

so that

$$t = \frac{1}{\sqrt{a}} \int_o^x \frac{dz}{\sqrt{(A^2 - z^2)\left[1 + \frac{b}{2a}(A^2 + z^2)\right]}}\ .$$

Let $z = A \sin y$, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. Then

$$t = \frac{1}{\sqrt{a}} \int_o^\psi \frac{A \cos y \, dy}{\sqrt{A^2 \cos^2 y\left[1 + \frac{b}{2a}(A^2 + A^2 \sin^2 y)\right]}}$$

$$= \frac{1}{\sqrt{a}} \int_o^\psi \frac{dy}{\sqrt{\frac{2a + bA^2}{2a}} \sqrt{1 + \frac{bA^2}{2a + bA^2} \sin^2 y}} \qquad \left(\psi = \mathrm{Sin}^{-1} \frac{z}{A}\right)$$

since $\cos y \geq 0$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Let

$$\mu = \frac{-bA^2}{2a + bA^2}$$

(note that $0 < \mu < 1$ since $0 < E_o = \frac{1}{2} a A^2 + \frac{1}{4} b A^4 < \frac{a^2}{-4b}$ with $0 < A < \sqrt{\frac{a}{-b}}$). Let

$$\omega = \sqrt{\frac{2a + bA^2}{2}} .$$

Then

$$\omega t = \int_o^{\psi} \frac{dy}{\sqrt{1 - \mu \sin^2 y}} = F(\Psi, \mu) .$$

Hence

$$x(t) = A \sin \Psi = A \text{ sn } F(\Psi, \mu)$$

or

$$x(t) = A \text{ sn}(\omega t; \mu) .$$

The generalization to permissible initial conditions

$$x(0) = x_o \in \left(-\sqrt{\frac{a}{-b}} , \sqrt{\frac{a}{-b}}\right),$$

$$\dot{x}(0) = v_o ,$$

$$0 < \frac{1}{2} v_o^2 + \frac{1}{2} a x_o^2 + \frac{1}{4} bx_o^4 < \frac{a^2}{-4b} ,$$

is exactly as before. By solving

$$\frac{1}{2} aA^2 + \frac{1}{4} bA^4 = E_o$$

for A (taking note that the larger possible value of A would give oscillations of such an amplitude $\left(>\sqrt{\frac{a}{-b}}\right)$ as to preclude periodicity) and by solving

$$A \operatorname{sn}(\alpha; \mu) = x_o$$

for α, a periodic solution to (1) satisfying the given initial conditions for the soft spring case is seen to be

$$x(t) = A \operatorname{sn}(\omega t + \alpha; \mu),$$

ω and μ as before. Thus the assumption of a periodic solution of (1) is justified by exhibition. That $x(t)$ as defined above does indeed satisfy (1) and the specified initial conditions is easily verified.

Comparison with the Linear Oscillator

That the period of free vibrations for a linear oscillator is independent of the initial conditions is well known. Such is not the case for a nonlinear oscillator, for consider the solution functions above. The period of the elliptic sine and cosine functions with modulus μ is $4K(\mu)$ where

$$K(\mu) = F\left(\frac{\pi}{2}, \mu\right) = \int_o^{\frac{\pi}{2}} \frac{dx}{\sqrt{1 - \mu \sin^2 x}}$$

is the complete elliptic integral of the first kind. Hence the period of the motion is

$$T_1 = \frac{4K(\mu)}{\omega_1} = \frac{4K(\mu)}{\sqrt{a + bA^2}}$$

for the hard spring and

$$T_2 = \frac{4K(\mu)}{\omega_2} = \frac{4K(\mu)}{\sqrt{\frac{2a + bA^2}{2}}}$$

for the soft spring (see Figures 2, 3).

Ordinarily the term ω as used here is called the angular frequency. However, it is worthy of note that in all such cases, the functions dealt with have constant period, usually 2π, as in the case of circular sinusoids. A second glance is perhaps then in order if ω is still to be called a frequency, for now there is no quantity M independent of the initial conditions and such that $\frac{\omega}{M}$ is the frequency of the motion (as is the case for circular functions, where $f = \frac{\omega}{2\pi}$ is the frequency of oscillation in cycles per unit time). Here the quantity M (2π for circular functions) must be replaced by $T = 4K(\mu)$, which depends on the initial conditions of the motion. Then for given initial conditions the frequency in cycles per unit time is

$$f = \frac{\omega}{4K(\mu)} .$$

For the hard spring it is seen that, for fixed a and b, as E_o increases, so does A, hence so does ω. It is also seen that $K(\mu)$ increases with A, so that the dependence of f on A is not clear. One notices however that $\mu = \dfrac{bA^2}{2(a + bA^2)}$, so that for fixed a and b, $0 < \mu < \frac{1}{2}$ for any initial conditions (any value of A). Hence, as A increases, $K(\mu)$ approaches a finite value, while ω increases

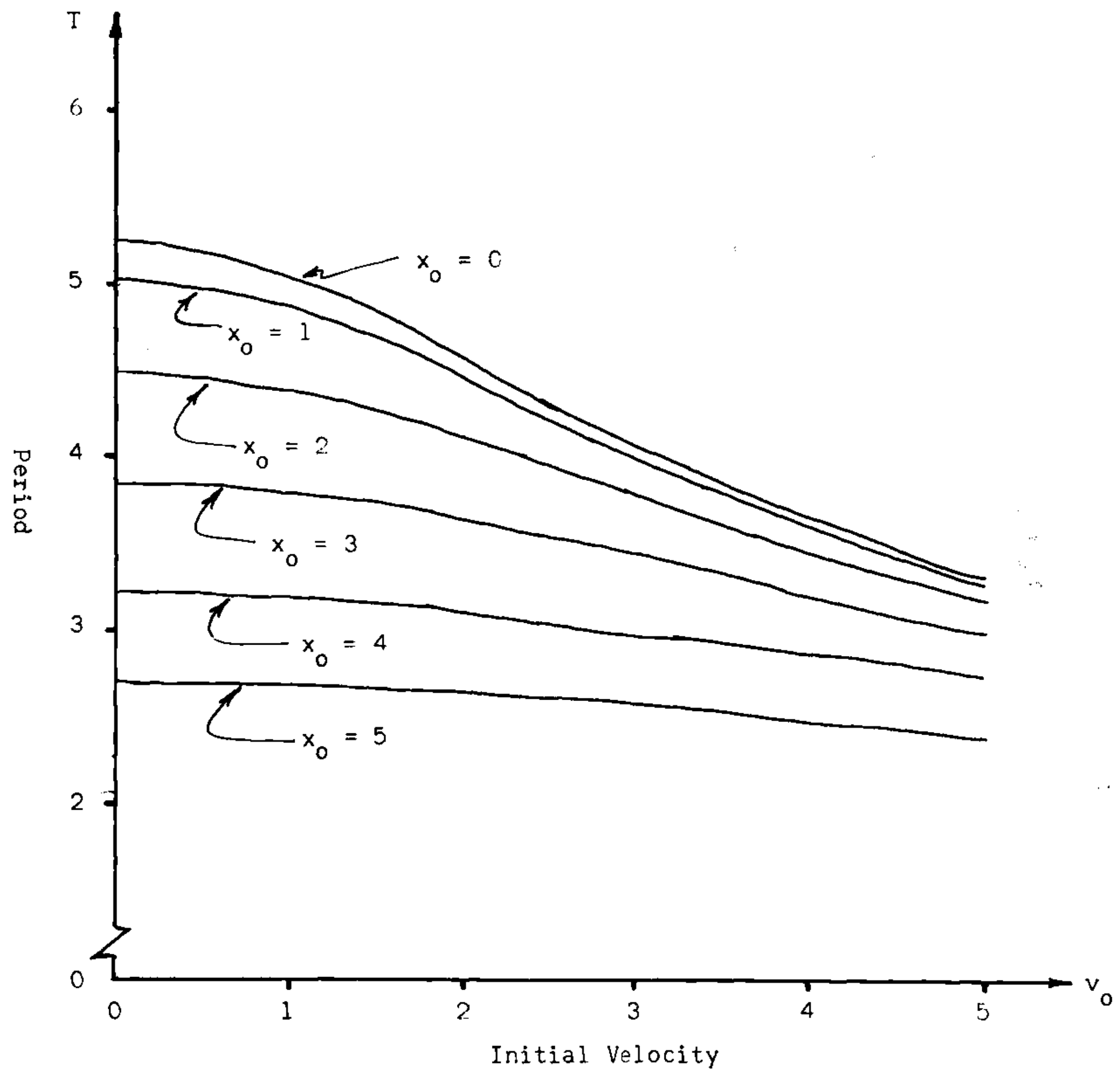
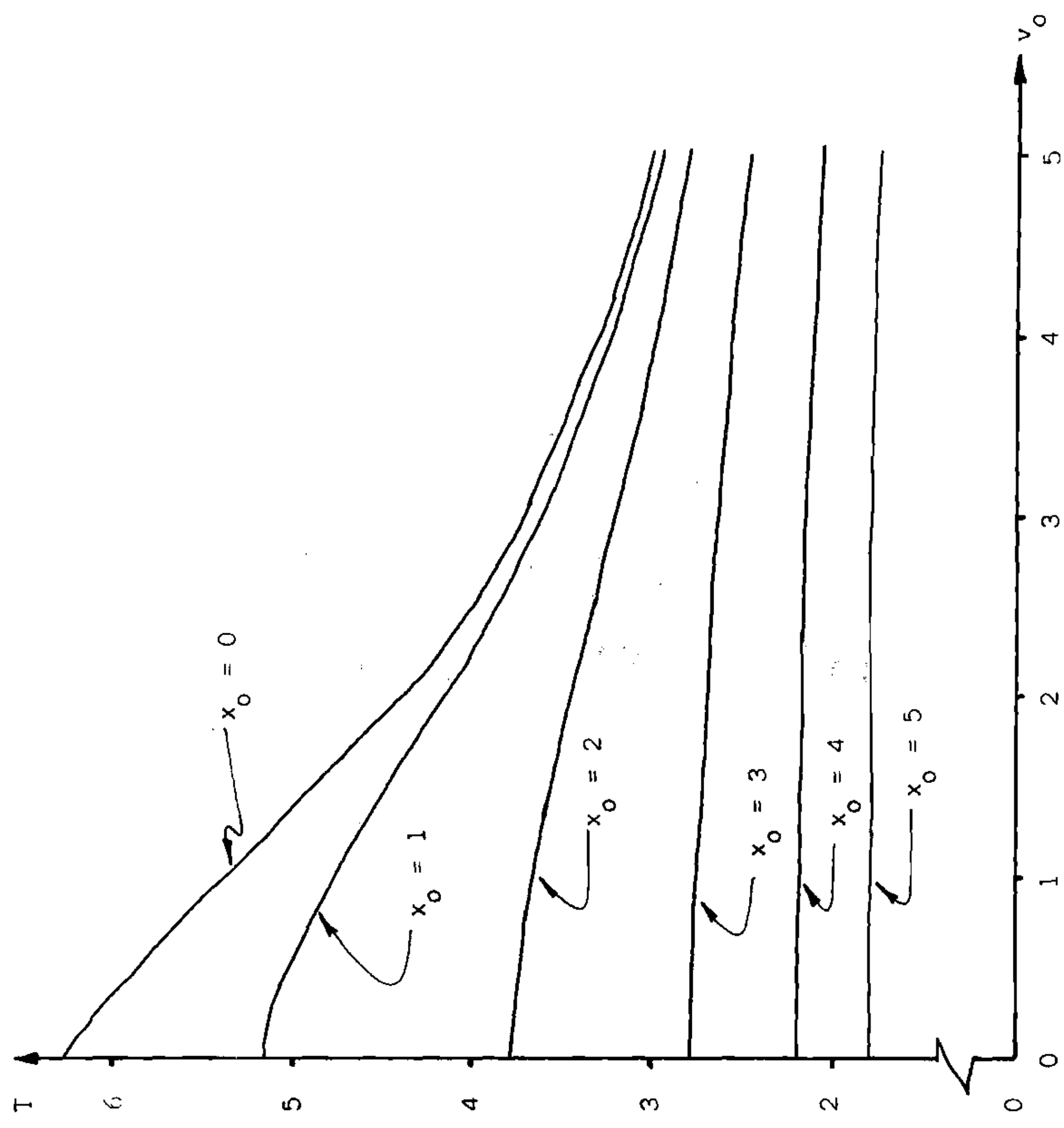


Figure 2. Period as a Function of Initial Velocity for Various Initial Displacements.
(See Figure 1(a), Hard Spring, a = 1, b = 0.1 m = 1.)



Period

Figure 3. Period as a Function of Initial Velocity for Various Initial Displacements. (See Figure 1(a), Hard Spring, a = 1, b=0.6, m = 1.)

without bound. Therefore, for the hard spring, cyclic frequency increases with amplitude.

For the soft spring, the opposite phenomena are observed. Since $b < 0$ in this case, $\omega^2 = \frac{2a + bA^2}{2}$ decreases with increasing A. The restrictions on system energy must be kept in mind when considering the changes in $K(\mu)$. In this case

$$\mu = \frac{-bA^2}{2a + bA^2},$$

$$A^2 = \frac{a - \sqrt{a^2 + 4bE_o}}{-b},$$

and

$$0 < E_o < \frac{a^2}{-4b}.$$

Hence

$$\mu = \frac{a - \sqrt{a^2 + 4bE_o}}{a + \sqrt{a^2 + 4bE_o}},$$

and as $E_o \rightarrow \frac{a^2}{-4b}$, $\mu \rightarrow 1$, $A^2 \rightarrow \frac{a}{-b}$, $\omega^2 \rightarrow \frac{a}{2}$, $K(\mu) \rightarrow +\infty$. Thus the cyclic frequency decreases with increasing amplitude for the soft-spring case, and as $A^2 \rightarrow \frac{a}{-b}$, $f \rightarrow 0$.

It is perhaps interesting to note that for system parameters a and b constant, the amplitude and hence frequency and period of the motion depend only on the initial energy E_o imparted to the system and are in no way dependent on what part of the initial energy is kinetic and what part is potential.

Figure 4 is a sketch of the displacement curves for the system discussed in this chapter for soft, linear, and hard springs. The model is for a unit mass, linear constant $a = 3$, cubic constant $b = \mp 0.25$, $x_o = 0$, $v_o = -4$. It should be noted that for these initial conditions $E_o = 8$, and $E_o = 9$ is the upper bound for periodicity when the spring is soft, so that the effect of the softness might be expected to be considerable. One might also note the very strong similarity of the curves for the nonlinear springs to sinusoids. For large values of μ, say $\mu > 0.5$, distortion of the elliptic functions from this sinusoidal appearance becomes more pronounced. However, for all hard springs, $\Omega < 0.5$; and even for soft springs, $\mu > 0.5$ implies "dangerous" proximity to $E_o = \frac{a^2}{-4b}$, so that the introduction of the cubic element to the spring seems to leave the wave shapes of the displacement curves largely unchanged over extended ranges of initial energy. For a discussion of these wave shapes, see Handbook of Mathematical Functions [5], Cunningham [1], and McLachlan [12].

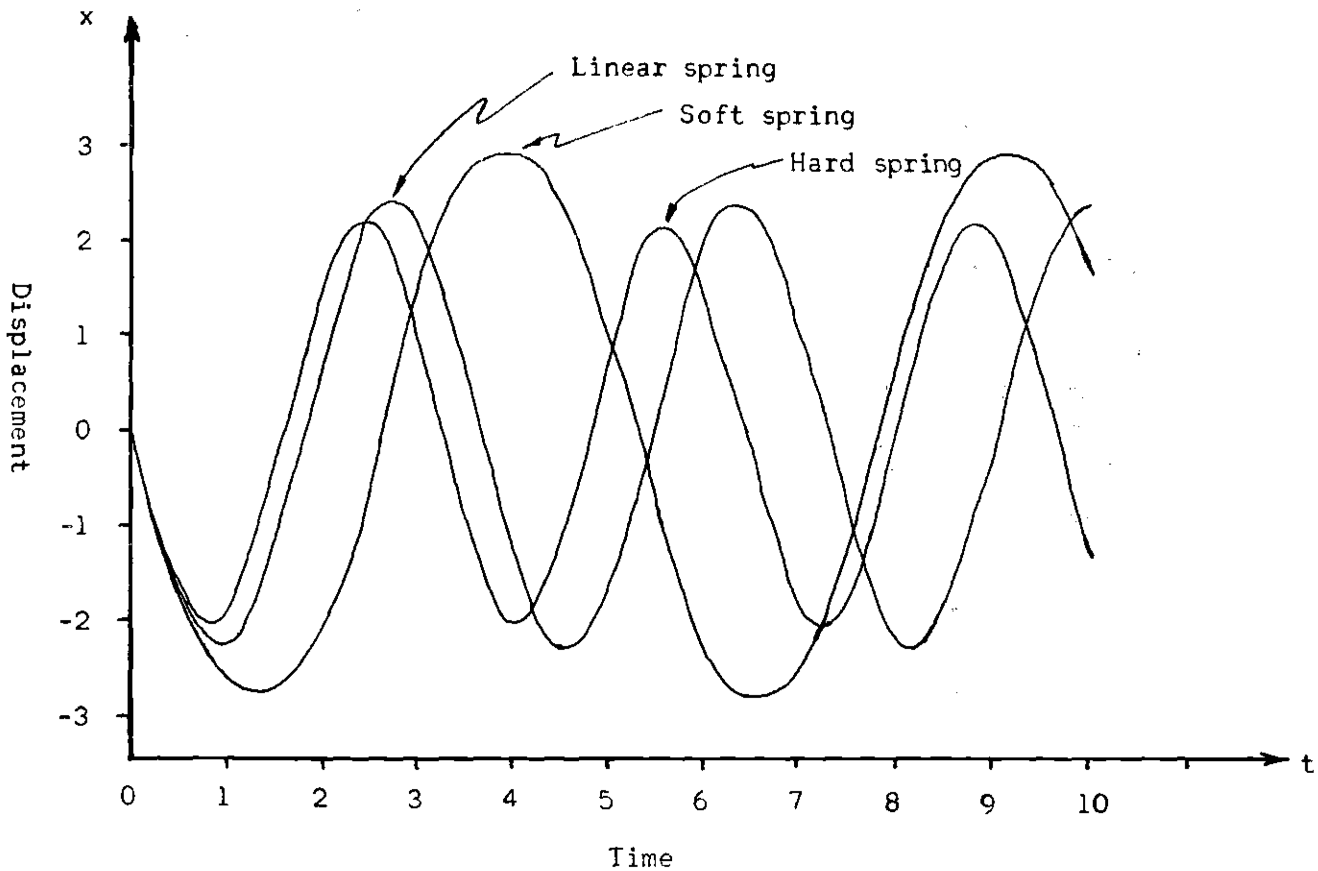


Figure 4. Displacement as a Function of Time for Soft, Linear, and Hard Springs.

CHAPTER IV

NONLINEAR MECHANICAL SYSTEMS WITH MORE THAN ONE DEGREE OF FREEDOM

In solving systems of coupled linear differential equations, the classical method is to assume an exponential solution and thus to reduce the problem to one of finding the eigenvalues of a matrix and the corresponding eigenfunctions. In vibration analysis this procedure is known as the process of finding the natural frequencies of vibration and the corresponding normal modes. Physically, this procedure amounts to finding the frequencies at which all elements can vibrate simultaneously, and then finding the dependent amplitude relations among the various coordinates.

An equivalent means of solving such linear systems is to use a similarity transformation to introduce coordinates so as to decouple the equations. The normal modes of the system are then obtained by setting respectively all but one of the new coordinates identically zero and solving the one remaining differential equation.

In considering vibrations of coupled nonlinear systems, phenomena similar to normal modes often appear. For example, if all of the springs of the system pictured schematically in Figure 5(a) are nonlinear of cubic characteristic with the same spring coefficients and if the masses are equal, it is easily shown that the in-phase and out-of-phase modes of vibration can occur as in the corresponding linear system. However, such vibrations can occur at many frequencies depending on the initial conditions,

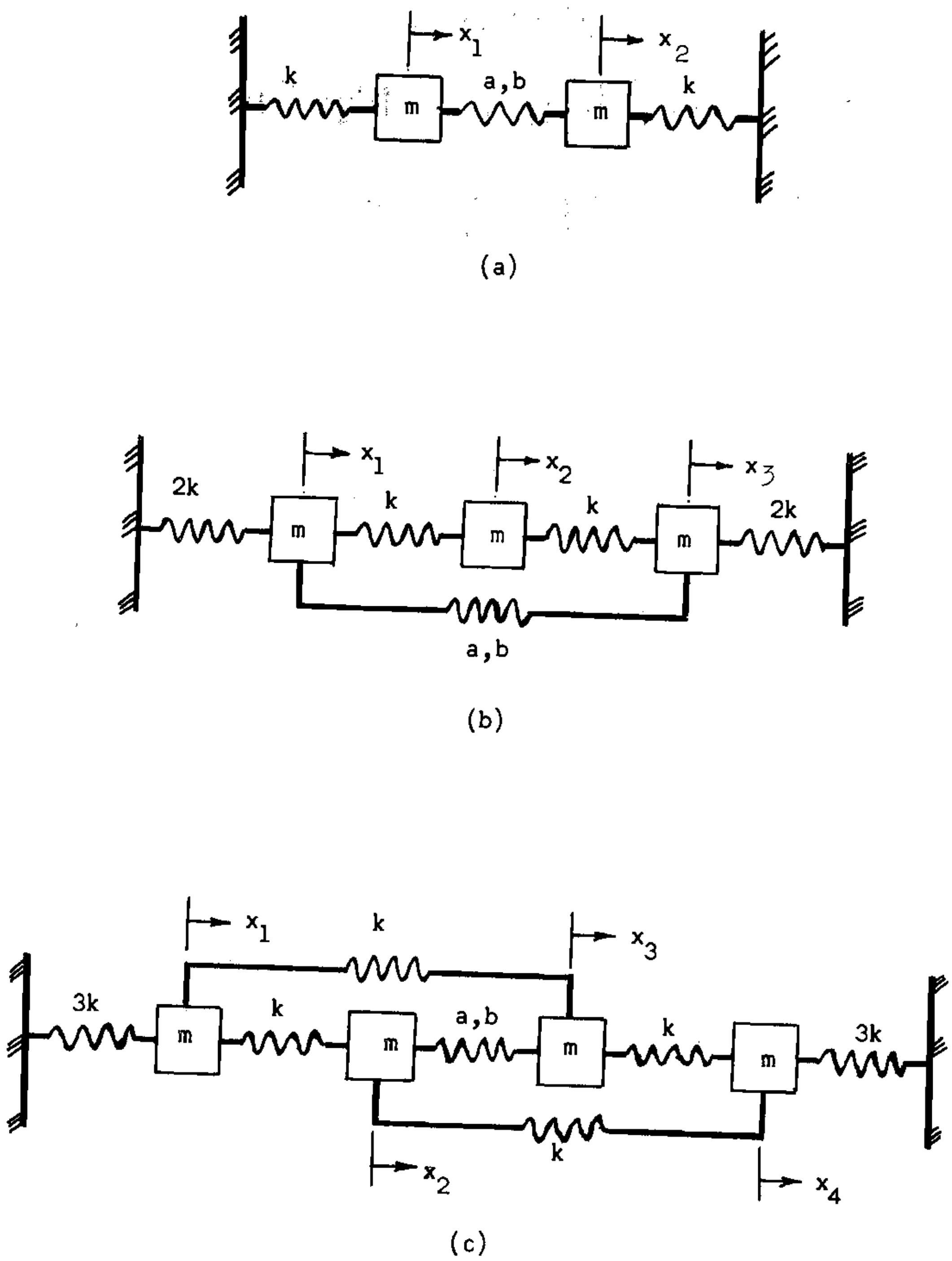


Figure 5. Mass-spring Systems with Several Degrees of Freedom.

a phenomenon never observed in linear systems. On the other hand such "normal modes" may not occur. For if the rightmost spring were missing in the above discussed configuration, the normal modes of the corresponding linear system would be described by relations of the form $x_1 = \lambda_1 x_2$ and $x_1 = \lambda_2 x_2$ for appropriate constants λ_1, λ_2. To show that no function x_1 and corresponding constant λ exist so that $x_1 = \lambda x_2$ is a solution of the corresponding nonlinear system is simple.

Using a definition of normal modes not entirely equivalent to that given in the preceding paragraphs, Rosenberg [8] has exhibited a nonlinear system possessing "normal-mode" phenomena not present in the corresponding linear system. An examination of his papers and of other studies makes clear that the discussion of normal modes of nonlinear systems is an involved subject in which even the definitions are not yet standardized.

One of the main reasons for studying normal modes of linear systems is the fact that any motion of such a system can be represented as a linear combination of these modes. Since the general superposition theorem is not true for nonlinear systems, a like result can hardly be expected in general. However, such results can be obtained in some instances.

Consider a coupled (nonlinear) differential system of N equations. Suppose a nonsingular linear transformation can be found which decouples the N equations. Then the i^{th} normal modal relation is defined to be the solution of the i^{th} uncoupled equation, the other $N-1$ transformed coordinates being identically zero. By finding these N normal modes and inverting the original transformation the original system is solved. It is to be noted that this is equivalent to the customary definition of normal modes for linear systems.

Nonlinear System with Two Degrees of Freedom

As a first example of this technique, consider the mass-spring system pictured schematically in Figure 5(a). Assume unit masses; let the outer springs be linear with constant k; and let the middle spring be nonlinear cubic with constants a, b. The coordinates x_1 and x_2 are to be measured from the equilibrium position of the masses. It should be noted that the springs are assumed to be at their natural lengths at equilibrium. Then the differential equations of motion for the system are

$$\left.\begin{aligned} \ddot{x}_1 &= -kx_1 + a(x_2 - x_1) + b(x_2 - x_1)^3 , \\ \ddot{x}_2 &= a(x_1 - x_2) + b(x_1 - x_2)^3 - kx_2 . \end{aligned}\right\} \tag{1}$$

Now make the linear transformation

$$\left.\begin{aligned} y_1 &= x_1 + x_2 \\ y_2 &= x_1 - x_2 . \end{aligned}\right\} \tag{2}$$

The system (1) thereby becomes

$$\ddot{y}_1 = - ky_1 , \tag{3a}$$

$$\ddot{y}_2 = - (k + 2a)y_2 - 2by_2^3 . \tag{3b}$$

Equation (3a) is readily seen to have solution

$$y_1(t) = A \cos \sqrt{k}\, t + B \sin \sqrt{k}\, t ,$$

where

$$A = y_1(0) = x_1(0) + x_2(0) \ ,$$

$$B = \frac{1}{\sqrt{k}} \dot{y}_1(0) = \frac{1}{\sqrt{k}} (\dot{x}_1(0) + \dot{x}_2(0)) \ .$$

By applying the analysis of Chapter III, the solution of (3b) when $b > 0$ is seen to be

$$y_2(t) = F \operatorname{cn} (\omega t + \varphi; \mu) \ ,$$

where

$$(2b)F^4 + 2(k + 2a)F^2 - 4E_o = 0$$

(E_o to be interpreted here as the initial energy of this mode; that is,

$$E_o = \frac{(\dot{y}_2(0))^2}{2} + \frac{k + 2a}{2} (y_2(0))^2 + \frac{(2b)}{4} (y_2(0))^4) \ ,$$

$$\omega^2 = (k + 2a) + 2bF^2 \ ,$$

$$\mu = \frac{(2b)F^2}{2[(k + 2a) + 2bF^2]} \ ,$$

$$y_2(0) = F \operatorname{cn} (\varphi; \mu) \ .$$

Similarly, if $b < 0$, the solution of (3b) is

$$y_2(t) = F \operatorname{sn}(\omega t + \varphi; \mu) \ ,$$

where F is the smaller positive zero of the equation

$$(2b)F^4 + 2(k + 2a)F^2 - 4E_o = 0 ,$$

E_o as before,

$$\omega^2 = (k + 2a) + bF^2 ,$$

$$\mu = \frac{-bF^2}{(k + 2a) + bF^2} ,$$

$$y_2(0) = F \text{ sn } (\varphi; \mu) .$$

It is to be noted that when $b < 0$, it is assumed that $0 \leq E_o \leq (k + 2a)^2/(-8b)$ in order to assure oscillatory motion.

The inverse of the transformation (2) is readily seen to be

$$\left.\begin{aligned} x_1 &= \frac{1}{2} (y_1 + y_2) \\ x_2 &= \frac{1}{2} (y_1 - y_2) . \end{aligned}\right\} \qquad (4)$$

By applying this inverse transformation to solve for x_1, x_2,

$$x_1(t) = \frac{1}{2} [A \cos \sqrt{k}\, t + B \sin \sqrt{k}\, t + F \text{ cn } (\omega t + \varphi; \mu)] ,$$

$$x_2(t) = \frac{1}{2} [A \cos \sqrt{k}\, t + B \sin \sqrt{k}\, t - F \text{ cn } (\omega t + \varphi; \mu)]$$

if $b > 0$, with the obvious change for $b < 0$.

Perhaps worthy of note here is the fact that the normal modes of the corresponding linear system can be obtained by the same decoupling transformation.

In the event that $b = 0$ above, the system is linear. The normal modes of the system are then

$$\text{(a)} \quad x_1 = x_2 = A \cos\sqrt{k}\, t$$

and $$\text{(b)} \quad x_1 = -x_2 = B \cos\sqrt{k + 2a}\, t$$

(note that if $b = 0$, then $\mu = 0$ for the hard- and soft-spring cases, and the elliptic functions degenerate to circular functions). Mode (a) corresponds to in-phase motion, while mode (b) corresponds to the higher frequency out-of-phase motion.

Consider the nonlinear system (1) subject to the initial conditions

$$x_1(0) = x_2(0) = A ,$$

$$\dot{x}_1(0) = \dot{x}_2(0) = 0 .$$

Then

$$x_1(t) = x_2(t) = A \cos\sqrt{k}\, t .$$

This is the in-phase mode and is exactly the same as the in-phase mode for the corresponding linear system. Such might be expected since only the coupling element is nonlinear, and it is not active for this motion.

Now suppose that the initial conditions for (1) are

$$x_1(0) = -x_2(0) = B ,$$

$$\dot{x}_1(0) = \dot{x}_2(0) = 0 .$$

Then if $b > 0$ (hard spring),

$$x_1(t) = -x_2(t) = B \text{ cn } (\omega t; \mu) ;$$

and if $b < 0$,

$$x_1(t) = -x_2(t) = B\ \mathrm{sn}\ (\omega t; \mu)\ ,$$

ω, μ as above for the respective cases. This is the out-of-phase motion which, in the linear case, has angular frequency $\omega_L = \sqrt{k + 2a}$. The angular frequency for the hard spring is

$$\omega = \sqrt{(k + 2a) + (2b)B^2} > \sqrt{k + 2a} = \omega_L ,$$

as might be expected since stiffening a linear spring increases angular frequency. For the soft spring,

$$\omega = \sqrt{(k + 2a) + bB^2} < \sqrt{(k + 2a)} = \omega_L$$

since $b < 0$, as again might be expected.

Hence, in at least this one nonlinear system, normal modes can be defined in complete analogy with the corresponding linear system and with the property that for any initial conditions which result in periodic motion the corresponding solution can be expressed as a linear combination of the normal modes.

Nonlinear System With Three Degrees of Freedom

As an example of a system of three degrees of freedom displaying properties similar to those described above, consider the mass-spring system pictured schematically in Figure 5(b). Assume unit masses. Let the two outer springs be linear with constant $2k$, the inner coupling springs connecting masses one to two and two to three be linear with constant k, and the upper coupling spring be nonlinear with cubic characteristic (coefficients a,b as before). This system will be solved completely only for the hard-spring case, with the obvious changes and restrictions for the soft spring. The equations of motion for the system are

$$\left.\begin{aligned} \ddot{x}_1 &= -2kx_1 + k(x_2 - x_1) + a(x_3 - x_1) + b(x_3 - x_1)^3, \\ \ddot{x}_2 &= -k(x_2 - x_1) + k(x_3 - x_2), \\ \ddot{x}_3 &= -k(x_3 - x_2) - 2kx_3 + a(x_1 - x_3) + b(x_1 - x_3)^3. \end{aligned}\right\} \tag{5}$$

By use of the linear transformation

$$\left.\begin{aligned} y_1 &= x_1 + 2x_2 + x_3 \\ y_2 &= x_1 - x_2 + x_3 \\ y_3 &= x_1 - x_3, \end{aligned}\right\} \tag{6}$$

(5) reduces to the uncoupled system of equations

$$\left.\begin{aligned} \ddot{y}_1 &= -ky_1, \\ \ddot{y}_2 &= -4ky_2, \\ \ddot{y}_3 &= -(3k+2a)y_3 - 2by_3^3, \end{aligned}\right\} \tag{7}$$

and each of these equations can be solved. In fact

$$y_1(t) = A \cos \sqrt{k}\, t + B \sin \sqrt{k}\, t,$$

$$y_2(t) = C \cos 2\sqrt{k}\, t + D \sin 2\sqrt{k}\, t,$$

and

$$y_3(t) = F \operatorname{cn}(\omega t + \varphi; \mu),$$

where

$$A = y_1(0), \quad B = \frac{1}{\sqrt{k}} \dot{y}_1(0),$$

$$C = y_2(0), \quad D = \frac{1}{2\sqrt{k}} \dot{y}_2(0) ,$$

$$(2b)F^4 + 2(3k + 2a)F^2 - 4E_o = 0 ,$$

$$E_o = \frac{1}{2}(\dot{y}_3(0))^2 + \frac{1}{2} (3k + 2a)(y_3(0))^2 + \frac{1}{4}(2b)(y_3(0))^4 ,$$

$$\omega = \sqrt{(3k + 2a) + (2b)F^2} ,$$

$$\mu = \frac{bF^2}{[(3k + 2a) + 2bF^2]} ,$$

$$F \text{ cn } (\varphi; \mu) = y_3(0) .$$

The inverse of the linear transformation represented by (6) is

$$\left.\begin{aligned} x_1 &= \frac{1}{6} (y_1 + 2y_2 + 3y_3) \\ x_2 &= \frac{1}{3} (y_1 - y_2) \\ x_3 &= \frac{1}{6} (y_1 + 2y_2 - 3y_3) . \end{aligned}\right\} \qquad (8)$$

Hence the system is easily recoupled and the solution is seen to be of the form

$$x_1(t) = \frac{1}{6} (X_1 \sin (\sqrt{k}\, t + \alpha_1) + 2X_2 \sin(2\sqrt{k}\, t + \alpha_2)$$
$$+ 3F \text{ cn } (\omega t + \varphi;\ \mu)) ,$$

$$x_2(t) = \frac{1}{3} (X_1 \sin (\sqrt{k}\, t + \alpha_1) - X_2 \sin (2\sqrt{k}\, t + \alpha_2)),$$

$$x_3(t) = \frac{1}{6}\,(X_1 \sin(\sqrt{k}\,t + \alpha_1) + 2X_2 \sin(2\sqrt{k}\,t + \alpha_2)$$
$$-3F\,\mathrm{cn}\,(\omega t + \varphi; \mu))\ .$$

Nonlinear System with Four Degrees of Freedom

As a final example of this type of nonlinear system, consider the mass-spring system of four degrees of freedom pictured schematically in Figure 5(c). The innermost spring, connecting masses two and three, is nonlinear with cubic characteristic (linear and cubic coefficients respectively a and b). The outermost springs are linear with constant $3k$. The others are linear with constant k. The equations of motion of the system are then

$$\left.\begin{aligned}
\ddot{x}_1 &= -3kx_1 + k(x_2 - x_1) + k(x_3 - x_1)\ , \\
\ddot{x}_2 &= -k(x_2 - x_1) + a(x_3 - x_2) + b(x_3 - x_2)^3 \\
&\qquad + k(x_4 - x_2)\ , \\
\ddot{x}_3 &= a(x_2 - x_3) + b(x_2 - x_3)^3 + k(x_4 - x_3) \\
&\qquad + k(x_1 - x_3)\ , \\
\ddot{x}_4 &= k(x_3 - x_4) - 3kx_4 + k(x_2 - x_4)\ .
\end{aligned}\right\} \qquad (9)$$

As in the example above, the system will be solved only for the hard spring, leaving the soft-spring case as an exercise for the reader.

The decoupling transformation for the system (9) is

$$\left.\begin{aligned}
y_1 &= x_1 + 2x_2 + 2x_3 + x_4 \\
y_2 &= -2x_1 + x_2 + x_3 - 2x_4 \\
y_3 &= x_1 - x_4 \\
y_4 &= x_2 - x_3\ .
\end{aligned}\right\} \qquad (10)$$

When (9) is decoupled by (10), the result is

$$\left.\begin{aligned}\ddot{y}_1 &= -ky_1 , \\ \ddot{y}_2 &= -6ky_2 , \\ \ddot{y}_3 &= -5ky_3 , \\ \ddot{y}_4 &= -2(a+k)y_4 - 2b\,y_4^3 .\end{aligned}\right\} \tag{11}$$

Solving for the normal modes yields

$$y_1(t) = A_1 \sin\sqrt{k}\,t + B_1 \cos\sqrt{k}\,t ,$$

$$A_1 = \frac{1}{\sqrt{k}}\,\dot{y}_1(0), \quad B_1 = y_1(0);$$

$$y_2(t) = A_2 \sin\sqrt{6k}\,t + B_2 \cos\sqrt{6k}\,t ,$$

$$A_2 = \frac{1}{\sqrt{6k}}\,\dot{y}_2(0), \quad B_2 = y_2(0);$$

$$y_3(t) = A_3 \sin\sqrt{5k}\,t + B_3 \cos\sqrt{5k}\,t ,$$

$$A_3 = \frac{1}{\sqrt{5k}}\,\dot{y}_3(0), \quad B_3 = y_3(0) ;$$

$$y_4(t) = F \operatorname{cn}(\omega t + \varphi; \mu) ,$$

$$2bF^4 + 4(a+k)F^2 - 4E_o = 0 ,$$

$$E_o = \frac{1}{2}(\dot{y}_4(0))^2 + (a+k)(y_4(0))^2 + \frac{b}{2}(y_4(0))^4 ,$$

$$\omega = \sqrt{2(a+k) + 2bF^2} ,$$

$$\mu = \frac{bF^2}{[2(a+k) + 2bF^2]} ,$$

$$F \operatorname{cn}(\varphi ; \mu) = y_4(0) .$$

The inverse of the decoupling transformation (10) is

$$\left.\begin{aligned} x_1 &= -\frac{1}{10}(y_1 - 2y_2 + 5y_3) \\ x_2 &= \frac{1}{10}(2y_1 + y_2 + 5y_4) \\ x_3 &= \frac{1}{10}(2y_1 + y_2 - 5y_4) \\ x_4 &= -\frac{1}{10}(y_1 - 2y_2 - 5y_3) \,. \end{aligned}\right\} \qquad (12)$$

Recoupling the system by (12) gives the solution of the system in the form

$$\left.\begin{aligned} x_1(t) &= -\frac{1}{10}[X_1 \sin(\sqrt{k}\,t + \alpha_1) \\ &\quad - 2X_2 \sin(\sqrt{6k}\,t + \alpha_2) + 5X_3 \sin(\sqrt{5k}\,t + \alpha_3)], \\ x_2(t) &= \frac{1}{10}[2X_1 \sin(\sqrt{k}\,t + \alpha_1) \\ &\quad + X_2 \sin(\sqrt{6k}\,t + \alpha_2) + 5F\,\mathrm{cn}(\omega t + \varphi;\, \mu)], \\ x_3(t) &= \frac{1}{10}[2X_1 \sin(\sqrt{k}\,t + \alpha_1) \\ &\quad + X_2 \sin(\sqrt{6k}\,t + \alpha_2) - 5F\,\mathrm{cn}(\omega t + \varphi;\, \mu)]\,, \\ x_4(t) &= -\frac{1}{10}[X_1 \sin(\sqrt{k}\,t + \alpha_1) \\ &\quad - 2X_2 \sin(\sqrt{6k}\,t + \alpha_2) - 5X_3 \sin(\sqrt{5k}\,t + \alpha_3)]. \end{aligned}\right\} \qquad (13)$$

A comment perhaps worthy of note here is that the decoupling transformation for each of the above examples is the same transformation which decouples the corresponding linear system.

Discussion of the Preceding Examples

The above three examples provide illustrations of a class of non-linear systems which exhibit properties analogous to the corresponding linear systems. There are, however, two properties of these systems which differ rather sharply from the linear cases. First, in each of the examples, one of the normal-mode equations is solved by an elliptic function rather than a sinusoid. For small oscillations (and even for large oscillations of a hard spring), the presence of elliptic functions seems to be of little physical importance, since no really significant difference is noted between the wave shapes of the elliptic functions and those for pure sinusoids of the same period and amplitude. However, in the soft-spring case, the very presence of the elliptic function is related to the fact that there is an upper limit on the initial energy which may be imparted to the system if oscillatory motion is to result; and when this limit is approached sufficiently closely from below, the wave shape of the elliptic function is substantially different from that of a sinusoid. The second difference is that when the system vibrates in the normal mode described by the elliptic function, it can do so at an uncountable infinity of frequencies depending on the initial energy. An interesting consequence of this second difference is discussed in the following paragraph.

In the study of oscillations, the occurrence or absence of periodic motion is often of considerable interest. For example, for a linear configuration of two masses and three springs as in Figure 5(a), the normal modes of vibration are at angular frequencies of $\sqrt{k}$ and $\sqrt{3k}$ radians per unit time if all the springs have constant k and both masses are unity. Hence if both normal modes are excited, periodic motion is impossible, since for a nondegenerate sum of periodic functions to be periodic, their periods must be rational multiples of one another. However, if the coupling element is nonlinear as described above, periodicity of the system is quite possible, since the period of the high frequency (out-of-phase) mode is variable. Indeed, in the notation used above in the analysis of this example, periodicity of the system occurs if and only if $\frac{2\pi}{\sqrt{k}} = r\,\frac{4K(\mu)}{\omega}$ for some rational r. Similar analysis shows what initial conditions lead to periodicity for the systems of three and four degrees of freedom.

Heinbockel and Struble [3] have shown that periodicity of systems possessing a certain symmetry property E defined below is easily studied. Those of their results which are pertinent to the present discussion are summarized in the following paragraphs.

Consider the vector differential equation

$$\mathring{z} = f(z) \tag{14}$$

where z and f are n-dimensional column vectors. It is assumed that f is such that the equation has a unique solution through each point $(t_o, z_o) \epsilon E_1 \times E_n$.

The differential system (14) is said to possess symmetry property E with respect to any $n \times n$ constant matrix Q for which

$$f(Qz) = -Qf(z)$$

for all z.

The lethargic set of the system corresponding to Q, denoted L_Q, is the set of constant vectors ω so that $Q\omega = \omega$. That is, L_Q is the set of fixed points of Q or the eigenvector manifold of Q associated with eigenvalue 1. If a trajectory of (14) emanates from L_Q at $t = 0$

and reenters L_Q after an elapsed time $T > 0$, then the trajectory is said to be L_Q-normal.

The following theorem is from [3], where a proof may be found.

Theorem 1: Let the system

$$\mathring{z} = f(z) \tag{14}$$

possess symmetry property E with respect to Q. Then every L_Q-normal trajectory of (14) is a periodic solution.

It may happen that the system (14) possesses symmetry property E with respect to two matrices. Such is the case for each of the systems considered in this chapter. Under such circumstances, Heinbockel and Struble have the following result which tends to simplify matters. Again, the proof is to be found in [3].

Theorem 2: Suppose that (14) possesses symmetry property E with respect to Q_1 and Q_2. Suppose that Q_2 maps L_{Q_1} into itself, that is $Q_2(L_{Q_1}) \subset L_{Q_1}$. Then any trajectory of (14) which first intersects L_{Q_2} and later L_{Q_1} is a periodic solution.

Let us now turn our attention to finding these matrices Q and their lethargic sets for the systems previously considered.

First consider the system of two degrees of freedom. With $\mathring{x}_1 = v_1$ and $\mathring{x}_2 = v_2$, the equations of motion are, in first order form,

$$\left.\begin{aligned} \mathring{x}_1 &= v_1 , \\ \mathring{v}_1 &= -kx_1 + a(x_2 - x_1) + b(x_2 - x_1)^3 , \\ \mathring{x}_2 &= v_2 , \\ \mathring{v}_2 &= -kx_2 + a(x_1 - x_2) + b(x_1 - x_2)^3 , \end{aligned}\right\} \tag{15}$$

or in vector form

$$\mathring{z} = f(z)$$

where $z = (x_1, v_1, x_2, v_2)^T$,

$$f\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} \beta \\ -k\alpha + a(\gamma - \alpha) + b(\gamma - \alpha)^3 \\ \delta \\ -k\gamma + a(\alpha - \gamma) + b(\alpha - \gamma)^3 \end{pmatrix} .$$

This system possesses symmetry property E with respect to the two matrices

$$Q_1 = \text{diag } (1, -1, 1, -1)$$

and

$$Q_2 = \text{diag } (-1, 1, -1, 1) = -Q_1 .$$

The lethargic sets corresponding to these matrices are readily seen to be composed of vectors of the form

$$z = (x_{10}, 0, x_{20}, 0)^T$$

for the matrix Q_1 and

$$z = (0, v_{10}, 0, v_{20})^T$$

for the matrix Q_2. Hence, for instance, if either both displacements or both velocities are simultaneously zero for distinct times t_1, t_2, then the motion is periodic by Theorem (1). It is also noted that $Q_2Q_1 = -I_4$. This implies that $Q_2(L_{Q_1}) = L_{Q_1} \subset L_{Q_1}$, so that if at some time t_1 both displacements are zero and at some subsequent time t_2 both velocities are zero, the motion is periodic by Theorem (2). The preceding statement is also true for velocities and displacements interchanged since $Q_1(L_{Q_2}) \subset L_{Q_2}$.

It is easy to show that the system of three degrees of freedom possesses symmetry E with respect to the two matrices

$$Q_1 = \text{diag } (1, -1, 1, -1, 1, -1)$$

and

$$Q_2 = \text{diag } (-1, 1, -1, 1, -1, 1) = -Q_1 ,$$

and the four-mass system with respect to

$$Q_1 = \text{diag } (1, -1, 1, -1, 1, -1, 1, -1)$$

and

$$Q_2 = \text{diag } (-1, 1, -1, 1, -1, 1, -1, 1) = -Q_1 .$$

The obvious extensions hold for the application of Theorems 1 and 2 to these larger systems.

It may seem that the examples presented in this chapter are very special and not very inclusive, due not only to their respective structures but to the choice of spring constants. This is in at least some degree true in that for many more general systems the normal modes defined herein do not exist, and hence such systems can hardly be expected to yield to the methods of solution used here. As to the choice of spring constants, so long as the obvious physical symmetry is preserved, the same linear transformations decouple the systems. One rather peculiar specialty which might be of some interest is that while in many examples only nearest-neighbor coupling is considered, here additional coupling of a straddling or non-nearest-neighbor nature had to be introduced before the systems could be decoupled.

CHAPTER V

SOME GRAPHICAL AND NUMERICAL RESULTS FOR THE TWO SIMPLEST SYSTEMS

Some phenomena of the systems considered thus far have been observed to be in strong contrast with the classical theory of linear systems. One of these is the fact that the period of one of the normal modes varies with initial conditions. It is also observed that while the elliptic functions describing the nonlinear normal mode degenerate into sinusoids as the nonlinearity goes to zero, they do exhibit properties which are analytically quite different from sinusoids.

As was noted in Chapter II, the periodic solutions of the basic nonlinear equation have phase-plane trajectories which bear strong resemblance to those of linear oscillators, whose solution functions are sinusoidal. This resemblance, together with the above mentioned fact that when nonlinearity disappears the elliptic functions degenerate into circular functions, might lead one to inquire as to the difference in wave shapes of elliptic and circular functions, at least for small nonlinearities. This difference will be seen to be small indeed.

To illustrate several of the phenomena encountered thus far, consider a numerical example. Let the one-dimensional oscillator discussed in Chapters II and III have unit mass, linear spring constant $a = 3$. Three cases will be considered: (1) hard spring, with cubic coefficient $b = 0.25$; (2) linear spring; (3) soft spring, with cubic coefficient $b = -0.25$. The initial conditions are $x(0) = 0$, $\dot{x}(0) = -4$. These are such as to give noticeable differences between the cases. Indeed, for the soft spring, $E_o = 8$ is nearly ninety per cent of the maximum $\frac{a^2}{-4b}$ for periodic motion, seen to be 9. The solutions of the differential equations of motion are then seen to be

$$x(t) = -\frac{4}{\sqrt{3}} \sin \sqrt{3}\, t$$

for the linear spring,

$$\begin{aligned} x(t) &= 2\sqrt{2} \ \text{sn}\ (\sqrt{2}\, t + 2K(0.5);\ 0.5) \\ &= 2\sqrt{2}\ \text{sn}\ (\sqrt{2}\, t\ ;\ 0.5) \\ &= -2.828\ \text{sn}\ (1.414t\ ;\ 0.5) \end{aligned}$$

for the soft spring, and

$$x(t) = 2.12\ \text{cn}\ (2.03t + 1.63\ ;\ 0.136)$$

for the hard spring. These displacements are sketched in Figure 4 for $0 \leq t \leq 10$.

As was seen analytically in Chapter III, the hard spring has the smaller amplitude while the soft spring has lower frequency. The similarity to sinusoids is evident. This leads one to suspect even more strongly that for a considerable range of the parameter μ there is little essential difference in the waveshapes of sinusoids and elliptic functions of the same amplitude and period.

A study of elliptic functions shows that if $f(t) = A \sin \omega t$ has period adjusted to that of $g(t) = A\ \text{sn}\ (\omega_1 t;\ \mu)$, then $|f(t)| \leq |g(t)|$ for all t, the equality occurring at extreme points and zeros. It is also seen that as μ increases toward 1, the sn function tends to be steeper near zeros and flatter near extrema, thus accentuating the inequality. However, for relatively small μ, say $\mu \leq 0.6$, the difference remains qualitatively and quantitatively rather small. It is also to be noted that the sn function describes the soft-spring oscillator and that $\mu = 0.6$ occurs when system energy is over ninety per cent of the maximum permissible.

It can also be shown that if $f(t) = A \cos \omega t$ and $g(t) = A \operatorname{cn}(\omega_1 t; \mu)$ have the same period, then $|g(t)| \leq |f(t)|$ for all t, equality occurring again only at zeros and extrema for $\mu > 0$. As μ tends to 1 from below, the difference becomes greater, g tending to become flatter near zeros and steeper near extrema. This behavior is seen geometrically to necessitate the introduction of an inflection point between each zero and either neighboring extremum. The point of inflection appears as soon as $\mu > 0.5$, thus destroying much resemblance between the cn and cos functions. But reference to Chapter III shows that for the hard spring, which is described by the cn, $0 < \mu < 0.5$. Hence the difference becomes accentuated only for extremely nonlinear springs, particularly for soft springs.

This phenomenon of flattening or sharpening at extrema can be seen to some extent in Figure 4, where the peaks seem to be sharper for the hard spring and rounder for the soft spring.

The conclusion that seems reasonably clear then is that over very wide ranges of nonlinearity, the motion induced by a cubic spring is described well by a sinusoid of appropriately adjusted amplitude and period.

To illustrate this conclusion, consider the two-mass coupled system described in Chapter IV. Assume unit masses. Let $k = 1$, $a = 1$, $b = \pm 0.125$ for the two cases of hard and soft springs. Let the initial conditions be $x_1(0) = x_2(0) = \dot{x}_1(0) = 0$, $\dot{x}_2(0) = 4$. The solutions for motion are then

$$x_1(t) = 2(\sin t + .53 \text{ cn } (2.03t + 1.63;\ 0.136))$$

$$x_2(t) = 2(\sin t - .53 \text{ cn } (2.03t + 1.63;\ 0.136))$$

for the hard spring, and

$$x_1(t) = 2 \sin t - \sqrt{2} \text{ sn } (\sqrt{2}\, t;\ 0.5)$$

$$x_2(t) = 2 \sin t + \sqrt{2} \text{ sn } (\sqrt{2}\, t;\ 0.5)$$

for the soft spring. These displacements are plotted in Figures 6 and 7. As a comparison, the elliptic functions in each case were replaced by a sinusoid of the same period, amplitude, and phase. The resulting curves were graphically indistinguishable for the hard spring and are plotted as points for the soft-spring displacements in Figure 7.

It is of some interest to note the distension of the coupling spring in this case. That this spring would undergo the same sort of pseudo-random oscillations as the two masses would seem very plausible -- perhaps almost self-evident. Such is not the case. The distension δ is given by $\delta(t) = x_1(t) - x_2(t)$ so that from the equations of motion, one sees that δ is periodic with the same period as the out-of-phase or high-frequency mode of oscillation. The equation for $\delta(t)$ is

$$\delta(t) = -2\sqrt{2} \text{ sn } (\sqrt{2}\, t\ ;\ 0.5)$$

for the soft spring, and

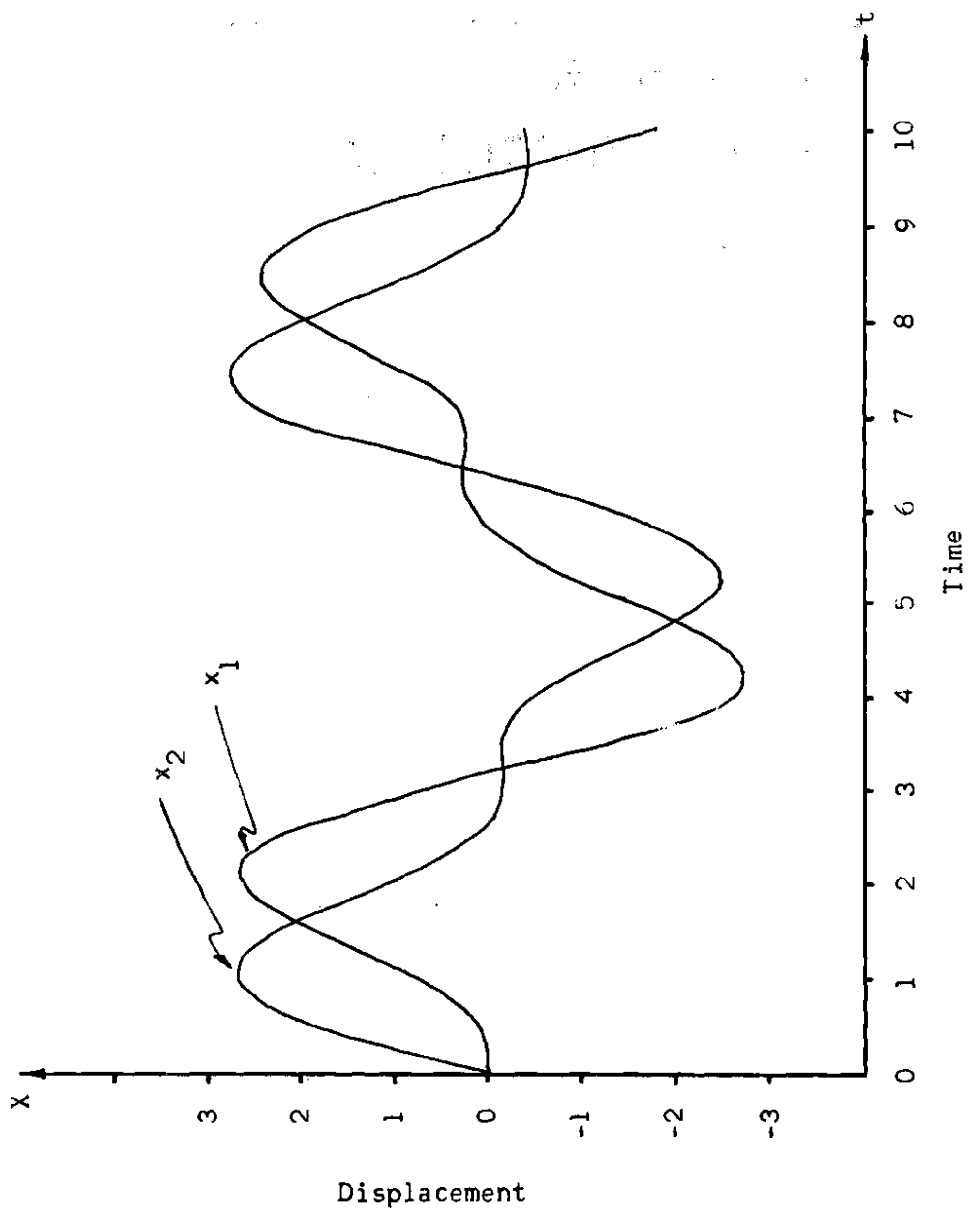


Figure 6. Displacements as Functions of Time for System of Two Degrees of Freedom with Hard Coupling Spring. (See Figure 5(a), a = 1, b = 0.125, k = 1, m = 1.)

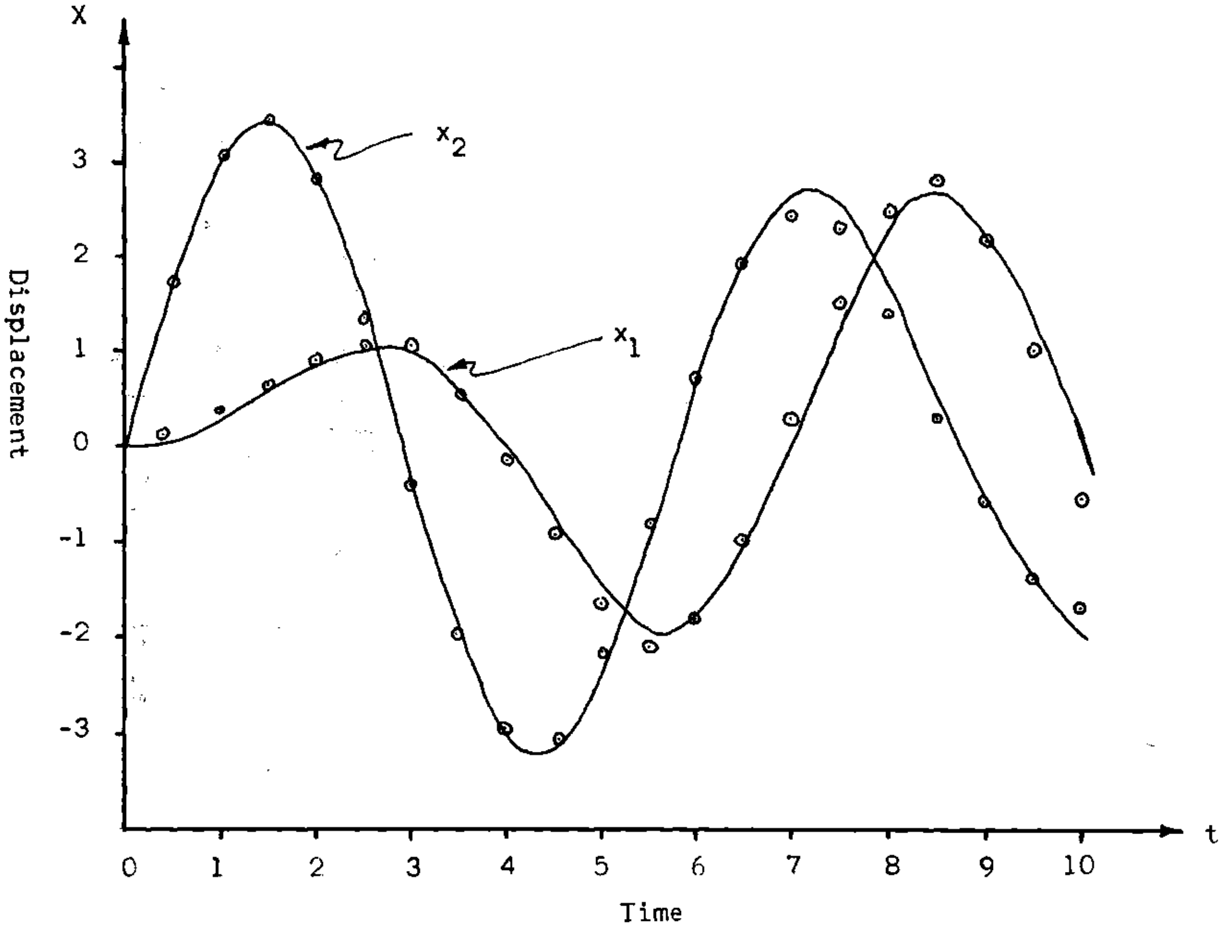


Figure 7. Displacements as Functions of Time for System of Two Degrees of Freedom with Soft Coupling Spring. (See Figure 5(a), a = 1, b = 0.125, k = 1, m = 1.)

$$\delta(t) = 2.12 \text{ cn } (2.03t + 1.63; 0.136)$$

for the hard spring. One might also note that the strain energy of this spring is given by

$$\varepsilon(t) = \frac{1}{2} a \, \delta^2(t) + \frac{1}{4} b \, \delta^4(t) ,$$

which is again periodic with half the period of the high-frequency mode of oscillation. Figure 8 is a sketch of $\varepsilon(t)$ vs t for the soft spring.

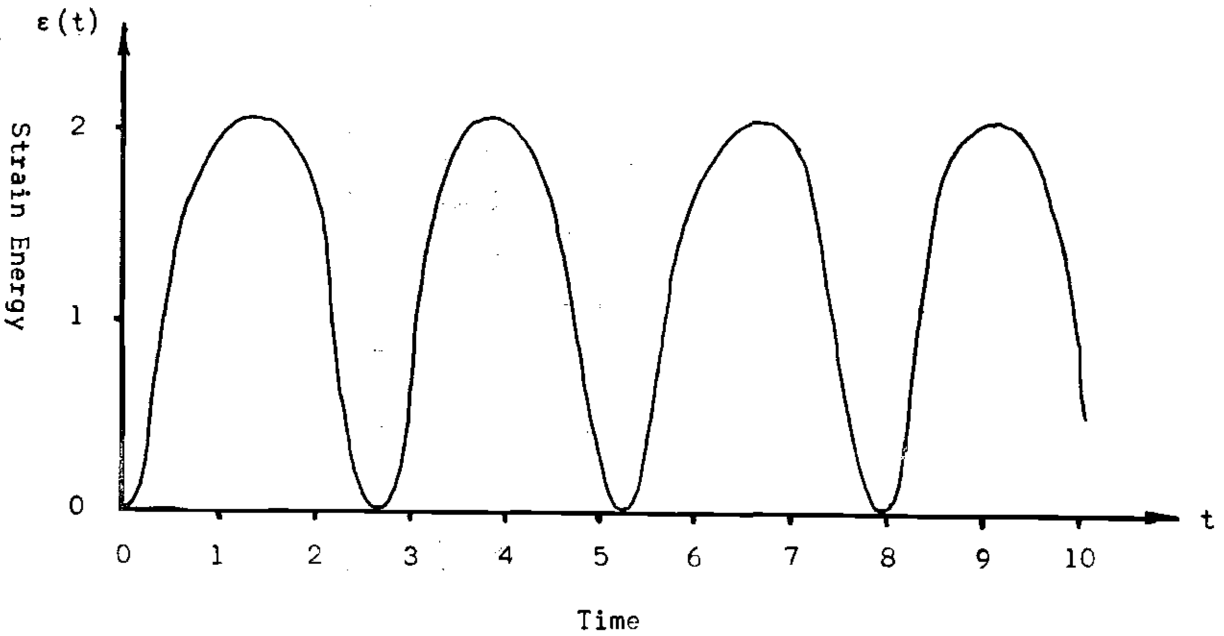


Figure 8. Strain Energy in Soft Coupling Spring. (See Figure 5(a), $a = 1$, $b = -0.125$, $k = 1$, $m = 1$.)

CHAPTER VI

SUMMARY OF PRINCIPAL RESULTS

In view of the discussion of the preceding chapters two conclusions seem clear about nonlinear systems of the type considered here: first, in some respects they differ drastically from the corresponding linear systems; and second, in other respects they are very similar to the corresponding linear systems.

As is readily seen, not all nonlinear systems yield to easy solution. The systems exhibited in this thesis, however, are easily solved by a direct extension of the classic method involved in the solution of linear systems. Namely, the system is decoupled by a linear transformation; the resulting transformed equations are solved for the normal modes as defined herein; and the general solution of the system is expressed as a linear combination of the normal modes. It is even seen that the behavior of the systems is quite similar to suitably adjusted linear systems over wide ranges of initial conditions, this because of the close proximity of wave shapes of sinusoids and elliptic functions for large ranges of the modulus μ.

Here the similarity ends and sharp differences come into focus, among which the following are of particular interest.

(1) Perhaps most obviously, in the nonlinear systems one normal mode is described by an elliptic function rather than a sinusoid.

(2) For a linear system each normal mode has a characteristic

frequency determined by the physical constants of the system and independent of the initial conditions. In each of the nonlinear systems considered, one normal mode can occur at many frequencies, the particular frequency present being determined by the initial conditions once the system parameters have been fixed. This dependence on the initial conditions implies that the frequency of the non-sinusoidal normal mode is a function of its amplitude.

(3) In a linear system, the amplitude is a quadratic in the initial conditions. For a nonlinear system with cubic characteristic, the amplitude is a quartic function of the initial coordinates.

(4) No matter how much initial energy is imparted to a linear system, each normal mode is sinusoidal. But for the nonlinear systems, one of the normal modes is not sinusoidal. In fact, for a soft spring, an excess of initial energy results in motion which is not even oscillatory.

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